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# Dynamical friction and instability of interface motion

Igor F Lyuksyutov

Institute for Theoretical Physics, Cologne University, D-50937 Cologne, Germany and Institute of Physics, 252028 Kiev, Ukraine

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**Abstract.** An elastic body moving in a quenched random force environment with a singular force correlator experiences an additional friction force which is *inversely* proportional to velocity. This *dynamical* friction mechanism is important, e.g., for a moving interface which is subjected to quenched random pinning forces. For sufficiently strong pinning forces, interface motion becomes unstable at values of driving force smaller than some critical one and the interface velocity drops. We expect in this case a first-order depinning transition with hysteresis in the dependence of the interface velocity on driving force. Instability in the interface motion remains under perturbations like an additional periodic driving force or thermal fluctuations. Interface motion instability can also take place in systems with small viscosity, when inertial effects are important.

## 1. Introduction

A number of transport phenomena involve the motion of an elastic medium subjected to quenched random forces. A well known example is a domain wall movement in a disordered magnet (see e.g. [1]). Other familiar examples include charge density wave (CDW) systems (see e.g. [2]), and fluid–fluid interface movement in a porous medium (see e.g. [3]). The driven motion of an interface in a medium with random pinning forces and the related problem of the nonlinear dynamic of sliding charge density waves (CDW) have been considered by many authors (for a review see e.g. [4, 5]). The basic equation describing the viscous motion of the $d$-dimensional interface can be written in the form

$$\mu^{-1}\frac{\partial z}{\partial t} = \gamma\nabla^2 z + f + \eta(x, z) \tag{1}$$

where $x$ is a $d$-dimensional vector describing the interface, $z$ is a interface coordinate in the direction normal to the interface, $\mu$ is a mobility, $f$ is a driving force, and $\eta(x, z)$ a random pinning force which acts on the interface. The latter has zero mean and correlator

$$\langle\eta(x, z)\eta(x', z')\rangle = \delta^d(x - x')\Delta(|z - z'|) \tag{2}$$

where $\langle ...\rangle$ means the average over the possible distributions of the random force $\eta(x, z)$. Though the details of random force correlations in the $x$ space are not important and one can use a simple white noise correlator, the behaviour of function $\Delta(z)$ near $z = 0$ is crucial for the interface problem [5]. One limitation is that the characteristic distance $a$ at which $\Delta(z)$ changes is finite.

The critical behaviour near the depinning threshold has been studied in detail in the limit of weak pinning force and viscous motion (for a review see e.g. [4, 5]). We briefly review the results. For steady interface motion at zero temperature the driving force $f$

should exceed some threshold force $f_c$. Near the threshold the interface velocity $v$ obeys a power law:

$$v \propto (f - f_c)^{\theta} \tag{3}$$

where the critical exponent $\theta$ is given by $\theta = 1 - \epsilon/9 + \mathcal{O}(\epsilon^2)$ [5]. Here $\epsilon = 4 - d$. It has been shown [4, 5] that singularity in the random force correlator in the $z$ direction ($\Delta(z)$) at $z = 0$ (see equation (2)) plays an important role in the interface depinning problem, namely (i) renormalizable theory requires nonzero $\partial\Delta(z)/\partial z \equiv \Delta'(z)$ at $z = 0$, and (ii) with nonzero $\Delta'(0)$ the threshold force $f_c \propto |\Delta'(0)|/\gamma$. If the initial correlator has $\Delta'(0) = 0$, then, according to [5], it will renormalize, and at some finite scale $\tilde{a}$ a singularity appear. As a result the threshold force $f_c \propto |\tilde{\Delta}'(\tilde{a})|/\gamma$, where $\tilde{\Delta}(z)$ is a renormalized correlator. One can also see the special role of singular $\Delta(z)$ in the mean-field approximation (see below), as has been found in [4, 6].

In contrast to the previous studies we consider here the interface motion in the random force environment in the high-velocity limit. We use the mean-field approximation which has been proved as a useful approximation to study the interface motion and has been used by many authors [4, 6–9]. The singular force correlator results in this case in the additional contribution to the friction force which is *inversely* proportional to the velocity. To analyse the problem we consider the probability distribution of the solutions of the stochastic differential equation (1). This probability obeys the Fokker–Planck equation. The solution of the Fokker–Planck equation shows the existence of the effective *dynamical* friction force which acts on the moving interface due to the rapid changes of the random force equation (2). Together with common viscous friction which grows with velocity, the dynamical friction can result in a minimum in the friction force at some critical velocity $v_c$. This minimum appears only for strong enough random force acting on an interface. When one decreases the driving force $f$ from the state with the velocity $v > v_c$ then at every $v \geqslant v_c$ the friction force $f_{fr}(v)$ decreases monotonically and can adjust to the driving force. But at $v < v_c$ the friction force starts to grow and exceed the driving force. This means that the interface motion is unstable at $v < v_c$ and the velocity can simply drop to zero.

In section 2 we consider the dynamical friction force and section 3 deals with instability in the interface motion.

## 2. Dynamical friction

Below we consider the influence of the quenched random potential on the moving interface. The interface velocity will be assumed high enough to be away from the critical region. It is natural to consider this finite-velocity case in the mean-field approximation. The mean-field model for the CDW problem has been formulated by Fisher (see [7, 4]). Different versions of the mean-field model have been used by many authors (see [8, 9, 6]) to study critical behaviour of the interface. In the mean-field approximation one describes the interface moving with velocity $v$ under the applied force $f$ by only one coordinate $z$ with the mean value $\langle z \rangle = vt$ (see [8, 9, 6]). The corresponding dynamical equation has the form

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -\gamma(z - vt) + f + \eta(z). \tag{4}$$

In the above we have put $\mu = 1$. The first term on the rhs in equation (4) describes the elasticity of the interface, the last one the random pinning force which acts on the interface. The latter has zero mean and correlator

$$\langle \eta(z)\eta(z') \rangle = \Delta(|z - z'|). \tag{5}$$

It is convenient to introduce relative displacement $h = z - vt$ with $\langle h\rangle = 0$. From equation (4) one has

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -\gamma h + f - v + \eta(h + vt). \tag{6}$$

From the formal point of view equation (4) describes a particle driven by a force $f$ via a spring with rigidity $\gamma$. This mechanical analogy of the mean-field model is schematically represented in figure 1.

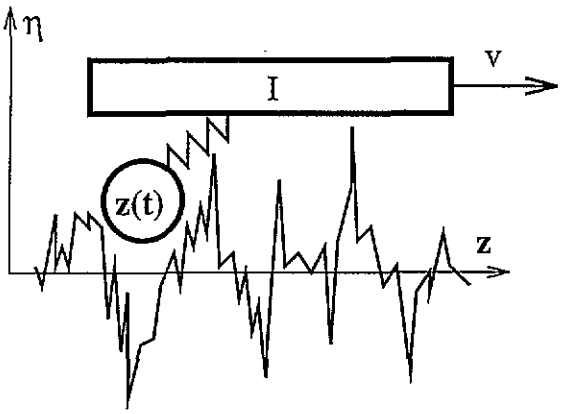


**Figure 1.** Schematic representation of the mean-field model. The particle with coordinate $z(t)$ is driven with velocity $v$ via a spring with rigidity $\gamma$. Block I represents schematically the mean position $\langle z\rangle = vt$ of the interface. $\eta(z)$ represents the random force which acts on the particle.

We consider the case of relatively high interface velocity. The high velocity means that the characteristic time $\tau_r = a/v$ at which the random pinning force changes due to the interface movement is much smaller then the typical relaxation time $\tau_i$ of the interface. In other words $v \gg a/\tau_i = v_i$. In the case of weak pinning force, namely $\Delta(0) \ll \gamma^2 a^2$, the upper limit on the time $\tau_i$ is $1/\gamma$. In the high-velocity limit one can average over the time $\tau$, such that $\tau_r \ll \tau \ll \tau_i$. To study the problem it is convenient to use methods from the theory of the stochastic differential equations. For every realization of the random pinning force one has some solution of equation (6). The distribution of these solutions is usually characterized (see [11, 10]) by probability $P(h', t|h_0, 0) \equiv P(h, t)$ where $h(t) = h'$, and $h(0) = h_0$. This probability distribution obeys the Fokker–Planck equation. One can derive this equation following the general procedure which is considered in the appendix (for references see e.g. [10, 11]). The Fokker–Planck equation for equation (6) has the form

$$\frac{\partial}{\partial t}P(h,t) = -\frac{\partial}{\partial h}P(h,t)A(h,t) + \frac{1}{2}\frac{\partial^2}{\partial h^2}P(h,t)B(h,t) \tag{7}$$

$$A(h,t) = -\gamma h + f - v + \frac{a}{v}A_0\Delta'(0) \tag{8}$$

$$B(h,t) = \frac{a}{v}B_0\Delta(0). \tag{9}$$

Calculation of $A(h,t)$ and $B(h,t)$ is found in the appendix, and $A_0$ and $B_0$ are constants of the order of unity. It is convenient to write $A(h,t)$ in the form $A(h,t) = \gamma(h_0 - h)$, where $\gamma h_0 = f - v + aA_0\Delta'(0)/v$. The last term in the expression for $h_0$ means that the moving interface can experience *dynamical* friction. This friction mechanism is effective only if $\Delta'(0) < 0$. For the Gaussian-like random force correlator in equation (5) $\Delta'(0) = 0$.

The simplest example of a random process with negative $\Delta'(0)$ is the Ornstein–Uhlenbeck process (see e.g. [10, 11]). It means that the random force $\eta(z)$ is a solution of the stochastic differential equation

$$\frac{\mathrm{d}\eta}{\mathrm{d}z} = -\frac{1}{a}\eta + \xi(z) \tag{10}$$

where $\xi(x)$ is a Gaussian white noise:

$$\left\langle \xi(z)\xi(z')\right\rangle = 2a^{-1}\Delta_0\,\delta(z-z'). \tag{11}$$

For the correlator equation (5) one has $\langle \eta(z)\eta(z')\rangle = \Delta_0 \exp(-|z-z'|/a)$ with $\Delta(0) = \Delta_0$, and $\Delta'(0) = \Delta_0/a$. The special role of the random forces with correlator with nonzero $\Delta'(0)$ has been recognized in [4, 5]. It has been found that in the limit of the weak pinning force ($\Delta(0) \ll \gamma^2 a^2$) one has finite threshold force $f_c$ only in the case of a pinning force with $\Delta'(0) < 0$ either at the smallest scale $a$ [4], or after renormalization at some finite scale $\tilde{a} > a$ ([5]). For the depinning (threshold) force one has $f_c \propto |\tilde{\Delta}'(\tilde{a})|/\gamma$, where $\tilde{\Delta}'(z)$ is a renormalized function $\Delta$. The mean-field model with such a pinning force correlator has been considered in [6]. The steady state solution of equation (7) has the form

$$P(h) = \frac{\sqrt{\gamma}}{\sqrt{\pi B}} \exp\left(-\frac{\gamma(h-h_0)^2}{B}\right). \tag{12}$$

To satisfy the condition $\langle h\rangle = 0$ one should put $h_0 = 0$. From the last condition one has the equation for interface velocity:

$$f - v - \frac{f_0^2}{4v} = 0 \tag{13}$$

where $f_0 = \sqrt{-4aA_0\Delta'(0)}$. Equation (13) has a simple physical meaning—in the case of steady movement of the interface one has two contributions to the friction force: one from the viscous friction which is proportional to $v$, and the other which is proportional to $(1/v)$, due to the singularity of the pinning force correlator. From (12, 13) for the case of a random force generated by the Ornstein–Uhlenbeck process one can find the lower limit on the diffusion constant $B = B_0 a\Delta(0)/v \leqslant 2aB_0\Delta(0)/f_0 = 2aB_0\Delta(0)/\sqrt{4aA_0|\Delta'(0)|} \approx a\sqrt{\Delta_0}$ and an upper limit on the mean square fluctuation of $h$: $\langle (h/a)^2\rangle \leqslant \sqrt{\Delta_0}/(\gamma a)$.

The dynamical friction force is not a specific feature of the interface mean-field model. One can see this from the fact that interface surface tension $\gamma$ does not enter equation (13). Important are relatively high velocity and singularity in the force correlator. The dynamical friction force has been found earlier for an energetic charge particle in an unmagnetized plasma (see e.g. [12, 11]). This problem was considered originally in [13]. Though the viscous movement of an interface and non-viscous flight of a particle are described by *different dynamical equations, the dynamical friction in both cases arises from the random* force environment.

In the case of the original model, equation (1), one can consider the interface as consisting of blocks connected by springs. At high interface velocity, fluctuations in the velocity of these blocks should be much smaller then the interface (or block) velocity. This means that one can apply the above consideration to single blocks with the same result—dynamical friction force.

## 3. Instability of interface motion

In the case of strong pinning force ($\Delta \gg \gamma^2 a^2$) dynamical friction has an important consequence for interface motion. For singular pinning force correlator ($\Delta'(0) < 0$) interface motion becomes unstable below some critical value of driving force. This conclusion follows from the solution of equation (13):

$$v = \tfrac{1}{2}(f_0 + \sqrt{f^2 - f_0^2}). \tag{14}$$

Equation (13) for interface velocity has a solution only for driving force $f \geqslant f_0$, and interface velocity is limited from below, namely $v \geqslant \frac{1}{2} f_0$. Qualitative explanation is straightforward—the friction force has a minimum value $f_0$ at $v_c = f_0/2$. With further decrease of the driving force the total friction force increases. This means that a driving force smaller then $f_0$ cannot hold the velocity at a value near $v_c$. As a result the velocity should drop down to some value $v_l$ which cannot be found by the above approach. Nevertheless one can obtain an upper limit on $v_l$. Indeed, the above consideration is restricted by the condition of high velocity, i.e. $v \gg \gamma a$, or $f_0 \gg \gamma a$. This condition coincides actually with the condition of strong disorder. For this reason in the limit of strong disorder, i.e. when $\Delta(0) \gg \gamma^2 a^2$, the threshold velocity $v_c = \frac{1}{2} f_0$ can be large enough ($v_c \gg v_i$) to satisfy the condition of the high-velocity limit ($\tau_r \ll \tau_i$). This means that with diminishing driving force the interface velocity will drop to zero or at least to the value $v_l \approx v_i = a/\tau_i \ll v_c$. In the region $v < v_i$ the above approach fails. For this reason we cannot rigorously prove that the velocity will drop exactly to zero. Nevertheless it seems the most probable scenario. One can support this result by comparison of $f_0$ with the value of threshold force obtained for the limit of low velocity. We know of detailed calculation of the threshold force for the original equation (1) only for the case of weak disorder (see [5]) when we expect a continuous transition. For this reason we use for comparison the mean-field result of Narayan and Fisher [4] for the so-called 'scalloped' potential. In this mean-field version the pinning force $\eta(z)$ is produced by the piecewise periodic potential with random amplitude, the so-called 'scalloped' potential. In this approach one can extend the analysis to the strong-potential limit. Dynamical friction is not detectable in this approximation. The corresponding expression for the threshold force has the form (equation (B.6b) in [4])

$$F_T = \pi \left\langle \frac{g^2}{(g+1)} \right\rangle \tag{15}$$

where variable $g$ describes the pinning force strength, and $\gamma = 1$. We have changed variable $h$ in the original work [4] to $g$. To compare with the above results consider the case of Ornstein–Uhlenbeck-like pinning force correlation. Then $\langle g^2 \rangle$ corresponds to $\Delta_0$, and in the limit of weak disorder ($\Delta_0 \ll \gamma^2 a^2$ or $\langle g^2 \rangle \ll 1$) $f_c \approx \Delta_0/\gamma$ which corresponds to $F_T = \pi \langle g^2 \rangle$, as follows from equation (15). According to equation (15) in the limit of strong disorder ($\Delta_0 \gg \gamma^2 a^2$ or $\langle g^2 \rangle \gg 1$) one has for $F_T = \pi \sqrt{\langle g^2 \rangle}$, which corresponds to $f_c \approx \sqrt{\Delta_0}$. This estimates support our suggestion that $f_c = f_0$.

In the case when the driving force increases from zero, one can expect that depinning proceeds via a first-order-like transition, i.e. with a velocity jump, in contrast to the second-order-like, continuous depinning transition in the case of weak disorder ($\Delta(0) \ll \gamma^2 a^2$). One can also expect that the interface starts to move at some $f_{c1} > f_0$, i.e. one has hysteresis in the interface velocity versus driving force dependence.

We have shown that in the limit of strong pinning force one has a velocity jump at the depinning transition. In contrast, in the weak-pinning-force limit one has a continuous transition. We expect that at the pinning force strength corresponding to $\Delta \approx \gamma^2 a^2$ one has a 'tricritical' point at which the continuous transition changes to a first-order transition.

### 3.1. Periodic driving force

Experimentally one can apply a periodic driven force to the CDW (see [14]). Consider the influence of an additional periodic driven force $A\cos\omega t$ on the above results. The equation of motion has the form

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -\gamma(z - vt) + f + A\cos\omega t + \eta(z). \tag{16}$$

By introducing the new variable $h = z - vt + b\cos\omega t + c\sin\omega t$ with $b = A\gamma^2/(\gamma^2 - \omega^2)$ and $c = A\gamma\omega/(\gamma^2 - \omega^2)$, one can rewrite equation (16) in the form

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -\gamma h + f - v + \eta((h + vt + b\cos\omega t + c\sin\omega t)/a) \tag{17}$$

In addition to the characteristic times $\tau_r$ and $\tau_i$ one has now the additional $\tau_\omega = a/c(\omega)\omega = a(\gamma^2 - \omega^2)/A\omega^2$. The corresponding Fokker–Planck equation is analogous to equation (7) with $A(h, t) = -\gamma(h - h_0)$, where $\gamma h_0 = f - v + aA_0\Delta'(0)/(v + c(\omega)\omega)$, and with $B(h, t) = aB_0/(v + c(\omega)\omega\Delta(0))$. By putting $x_0 = 0$ one obtains for the velocity

$$v = \tfrac{1}{2}(f - c(\omega)\omega) + \tfrac{1}{2}\sqrt{(f + c(\omega)\omega)^2 - f_0^2}.$$

This result means that at small values of $c(\omega)\omega$ the periodic force will not eliminate instability in the interface motion.

To study temperature effects one can introduce an additional thermal noise term $\eta_T(t)$ into equation (6) with correlator $\langle\eta_T(t)\eta_T(t')\rangle = 2T\,\delta(t - t')$. Thermal noise is additive to the quenched random force, has no influence on function $A(h, t)$ in equation (7) and changes only the value of diffusion constant $B(h)$, namely $B_T = B(h) + 2T$.

### 3.2. Small viscosity

Above we have considered the case of large viscosity when one can neglect the inertial term. Consider the mean-field interface model in the limit of small viscosity. We are mainly interested in the stability of the above results with respect to the viscosity decrease. In the limit of vanishing viscosity one should take into account the inertial term, namely, instead of equation (4) (we have put $\mu = 1$) one has:

$$m\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + \frac{\mathrm{d}z}{\mathrm{d}t} = -\gamma z + f + \eta(z). \tag{18}$$

Substituting $z = h + vt$, and $(\mathrm{d}h/\mathrm{d}t) = u$, one has

$$\frac{\mathrm{d}h}{\mathrm{d}t} = u \tag{19}$$

$$m\frac{\mathrm{d}u}{\mathrm{d}t} = -u - \gamma h + f - v + \eta(h + vt). \tag{20}$$

In addition to the relaxation times $\tau_i$ and $\tau_r$ one has the viscous relaxation time $\tau_v = m$. We will consider the situation when $\tau_r \ll \tau_i, \tau_v$. If $\tau_i \gg \tau_v$, then we have the viscous case, which has been considered above. The corresponding Fokker–Planck equation has the form

$$\frac{\partial P(h,u,t)}{\partial t} = -u\frac{\partial P(h,u,t)}{\partial h} + \frac{\gamma}{m}(h - h_0(f,v))\frac{\partial P(h,u,t)}{\partial u} + \frac{1}{m}\frac{\partial uP(h,u,t)}{\partial u} + \frac{B}{2m^2}\frac{\partial^2 P(h,u,t)}{\partial u^2} \tag{21}$$

where $(\gamma/m)h_0(f, v) = f - v + aA_0\Delta'(0)/v$, and $B = aB_0\Delta(0)/v$. The stationary solution of equation (21) is well known (see e.g. [10]):

$$P(h, u) = \exp\left(-\frac{\gamma}{B}(h - h_0)^2 - \frac{m}{B}u^2\right). \tag{22}$$

Using the condition $\langle h_0 \rangle = 0$ we have for $v$ an equation which coincides with equation (14) for the pure viscous case.

According to equation (22) the relative velocity $u$ fluctuates with mean square $\langle u^2 \rangle \approx a\Delta(0)/(vm)$. From the high-velocity condition $\tau_v \gg \tau_r$ or $m \gg a/v_0 \approx a/\sqrt{\Delta(0)}$ one has that $\langle u^2 \rangle / v^2 \leqslant a\Delta(0)/(v_0^3 m) \approx a/(m\sqrt{\Delta(0)}) \ll 1$. The last inequality means that under conditions of high velocity $v$, fluctuations of velocity are small and the mean-field approximation is valid. Analysis of the limit of vanishing viscosity show that this is no longer true when $\mu \to \infty$. We will analyse the limit $\mu \to \infty$ in a separate publication. We can conclude that under the above conditions one has the same physical picture of interface motion instability for the low-viscosity case as for purely viscous motion of an interface.

### 3.3. Experimental observation

We have found that, under some conditions, two phenomena which differ from the common picture of interface movement can take place: the dynamical friction, and, due to dynamical friction, the instability of interface motion. It will be interesting to find this type of behaviour in an experimental system.

Note that the dynamical friction force behaves analogously to the dry friction force i.e. it grows with velocity decrease (see e.g. [15]), though equation (1) describes the pure viscous motion. Such unusual behaviour of the friction force can be a useful guide in the search for corresponding experimental systems.

We have considered the interface in a mean-field approximation and have found the threshold velocity for the infinite interface. Below the threshold velocity the interface motion becomes unstable. On the finite scale the local 'threshold' velocity can vary significantly from one part of the interface to the other. This means that in a real system the instability manifests itself in the strong fluctuations in the velocity of different parts of the interface. As a result instead of a 'sharp' transition in the mean-field solution for an infinite system, in a finite system one should observe in some range near the threshold interface velocity strong irregularities and hysteresis phenomena in the interface motion.

The first-order depinning transition is possible under some conditions only. The strongest limitation is a necessity for a strong random force ($\Delta \gg \gamma^2 a^2$) with a singular correlator (see equation (2)) acting on the interface. One cannot expect that this condition can be realized for every domain wall or CDW system. Nevertheless, we hope that a first-order depinning transition will be found in experiment.

## 4. Conclusions

We have shown that elastic bodies (interface, charge density wave, etc) which move with sufficiently high velocity through media with quenched random forces which have a singular correlator experience a *dynamical* friction force which is *inversely* proportional to velocity. The influence of this dynamical friction force on interface dynamics depends on the relative strength of disorder, interface stiffness and viscosity. In the case of strongly quenched pinning forces this dynamical friction results in the instability of interface motion at values of driving force smaller than the critical force $f_c$. We expect in this case a first-order depinning transition with hysteresis in the dependence of interface velocity on driving force.

Instability in the interface motion remains under perturbations like an additional periodic driving force or thermal fluctuations. This instability can also take place in systems with small viscosity, when inertial effects are important.

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## Appendix. The Fokker–Planck equation

Below we briefly consider the derivation of the Fokker–Planck equation (7). The high-velocity limit means that the characteristic time $\tau_0 = a/v$ at which the random potential changes due to the interface movement is much smaller than the relaxation time $1/\gamma$. Then one can average over the time $\tau$, such that $\tau_0 \ll \tau \ll 1/\gamma$. The distribution of the possible solutions one can characterize by probability $P(h', t|h_0, 0) = P(h, t)$ to have, at time, $t$ $h(t) = h'$, when $h(0) = h_0$. This probability distribution obeys the Fokker–Planck equation of the form equation (7). One can derive this equation following the general procedure (see e.g. [10]). Functions $A(h, t)$ and $B(h, t)$ one can calculate from

$$A(h,t) = \lim_{\tau\to 0} \frac{1}{\tau}\Big\langle h(t+\tau) - h(t)\Big\rangle \tag{A1}$$

$$B(h,t) = \lim_{\tau\to 0} \frac{1}{\tau}\Big\langle (h(t+\tau) - h(t))^2\Big\rangle. \tag{A2}$$

The difference $h(t+\tau) - h(t) \equiv \Delta_\tau h$ can be represented as follows:

$$\begin{aligned}
\Delta_\tau h = \int_t^{t+\tau} \mathrm{d}\theta \Big\{-\gamma h(t+\theta) + f - v + \eta(h(t+\theta) + v(t+\theta))\Big\} \\
\approx (-\gamma h(t) + f - v)\tau + \int_t^{t+\tau} \mathrm{d}\theta\, \eta(h(t) + v(t+\theta)) \\
+ \int_t^{t+\tau} \mathrm{d}\theta\, \frac{\partial}{\partial h}\eta(h(t) + v(t+\theta))\Delta_\tau h \\
\approx (-\gamma h(t) + f - v)\tau + \int_t^{t+\tau} \mathrm{d}\theta\, \eta(h(t) + v(t+\theta)) \\
+ (-\gamma h(t) + f - v)\tau \int_t^{t+\tau} \mathrm{d}\theta\, \frac{\partial}{\partial h}\eta(h(t) + v(t+\theta)).
\end{aligned} \tag{A3}$$

After averaging equation (A3) over the possible distribution of the random forces one has $A(h, t) = -\gamma h + f - v + A_s$. For $A_s$ one has

$$\begin{aligned}
A_s = \lim_{\tau\to 0} \frac{1}{\tau}\left\langle \int_t^{t+\tau} \mathrm{d}\theta\, \frac{\partial}{\partial h}\eta(h + v\theta) \int_t^{t+\theta} \mathrm{d}\theta'\, \eta(h + v\theta')\right\rangle \\
= \lim_{\tau\to 0} \frac{1}{\tau} \int_t^{t+\tau} \mathrm{d}\theta \int_t^{t+\theta} \mathrm{d}\theta'\, \Delta'(v(\theta - \theta')) = \frac{a}{v} A_0 \Delta'(0)
\end{aligned} \tag{A4}$$

where $\Delta'(0) = (\partial\Delta(l)/\partial l)|_{l=0}$, and $A_0$ is a numerical constant of the order of unity. For $B(h, t) \equiv B$ one has

$$B = \lim_{\tau\to 0} \frac{1}{\tau}\left\langle \int_t^{t+\tau} \mathrm{d}\theta\, \eta(h + v\theta) \int_t^{t+\theta} \mathrm{d}\theta'\, \eta(h + v\theta')\right\rangle$$

$$= \lim_{\tau\to 0} \frac{1}{\tau} \int_t^{t+\tau} d\theta \int_t^{t+\theta} d\theta' \, \Delta(v(\theta-\theta')) = \frac{a}{v} B_0 \Delta(0) \qquad \text{(A5)}$$

and $B_0$ is a numerical constant of the order of unity.

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